THE PROBLEM OF DIVISION INTO "EXTREME AND MEAN RATIO"

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ABSTRACT

In our work we discuss the problem of division into "extreme and mean ratio" in terms of historical origin and evolution. Initially, we refer to the our study related proposals of Books II, IV, VI and XIII of the Euclid's Elements and present the evidence of two main proposals of the problem (II.11 and VI.30). Furthermore, we examine the relationship of the properties of this ratio according to the Golden Section problem. Finally, we are looking for when and where the Golden Section is used and under which prism.

INTRODUTION

" Geometry has two great treasures.
One is the theorem of Pythagoras, the other is the division of a line into extreme and mean ratio (Golden section).
The first we may compare to a measure of gold, the second we may name a precious jewel."
Johannes Kepler

THE PROBLEM

The origin is not historically verified. Some historians attribute it to the Pythagoreans and connect it to the study of the equation $x^2+ax= a^2$ as shown in geometric language in the Book II of Euclid's Elements , or the discovery of asymmetry in Ancient Greece, and others connect it with the construction of the pentagon from Theaetitos (415 π.Χ.-369 B.C.).

The division of a line AB into two segments using a point C so that, the large segment AC and the small segment CB with the whole of the line AB form equal ratios it is known as golden ratio, divine ratio, golden mean, golden section and Phi (φ).

$$\frac{AB}{AC} = \frac{AC}{CB}$$

This terminology is similar to what the Euclid of Alexandria (≈300 BC) defines the division into "extreme and mean ratio" in Book VI of the Elements. The existence of the point that divides "a straight line" to "extreme and mean ratio" and the geometric division (construction), are described in the proposals II.11 and VI 30 of Books II and VI of the Elements of Euclid.

The problem of "division into middle and end ratio" that appears from Book II, IV, VI and XIII of Euclid's Elements is associated with 'parable passages', also appears on the construction of a pentagon, icosahedron and dodecahedron and is used in proportions theory.

The concept of division in middle and end ratio did not end with the Euclid's Elements, but continued to play an important role in the development of mathematics and is directly related to the problem later in the 19th century, called the problem of the golden mean with which the equation is geometrically resolved, which positive root is related to the number

$$\varphi = (\sqrt{5} + 1)/2.$$

THE PROBLEM IN THE *ELEMENTS* OF EUCLID

Several terms are referred to the problem we are discussing. Therefore we present some initial useful terms related to the problem of our study as:

• A quantity x is called average ratio of two quantities a and b if the relationship $\frac{\alpha}{x} = \frac{x}{b}$ or equivalent $x^2 = \alpha \cdot b$ applies.

• The average proportional is constructed twice in the Elements of Euclid, in the proposal II.11 and the proposal VI.30, where it was first reported as mean and extreme ratio.

• Two segments a and b with a> b are in 'extreme and mean ratio' if $\frac{\alpha}{b} = \frac{b}{\alpha - b}$ or equivalent $\alpha^2 = \alpha \cdot b + b^2$.

In the second book of the Euclid's Elements is a set of proposals that are essentially geometric formalities of algebric formulas. The studied figures are always segments. Instead of the "product" we say "the rectangle contained by the a and b" and instead of α^2 we have "from the α square."

The proposals of the Elements associated with the problem of division into "extreme and mean ratio" are the II.11, VI.30, IV.10, IV.11, XIII. 16, XIII. 17 of Book II, IV, VI and XIII. From these we will present the evidence of proposals II.11 and VI.30 concerning the geometric construction of the subject of our study.

Proposition II.11 (1st construction): *To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment.*

Definition VI.3: *A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less.*

Proposition VI.30 (2st construction- golden section): *To cut a given finite straight line in extreme and mean ratio.*

Proposition IV.10: *To construct an isosceles triangle having each of the angles at the base double the remaining one. (golden triangle 72^0-72^0-36^0).*

Proposition IV.11: *To inscribe an equilateral and equiangular pentagon in a given circle.*

Proposition XIII. 16: *To construct an icosahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the square on the side of the icosahedron is the irrational straight line called minor.*

Proposition XIII. 17: *To construct a dodecahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the square on the side of the dodecahedron is the irrational straight line called apotome.*

In the proposal II.11 is referred the expression "extreme and mean ratio" because it relates immediately with the construction of regular pentagon inscribed in a circle of radius R, namely the division of the circle into five equal arcs [Proposals IV.10 and IV.11]. There, it is proved that the side and diagonal of regular pentagon is also a product of self-similar golden mean. In the definition VI 3 is defined as the point that divides the straight line to end and average ratio. In proposal VI.30 now, again II.11 is proved in a construction way (as a special case of a parabola over hyperbola) and this proposal immediately in Book VII, is producing numbers.

In Book XIII properties of the mean and the average ratio are proved [Proposals XIII.1 to XIII.6] because the icosahedron and a dodecahedron are related to regular pentagon.

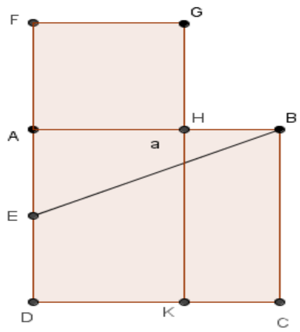
Proposition II. 11 (1st construction)

To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment.

Proof

We are asking to find point H of AB so that $AB \cdot HB = AH^2$

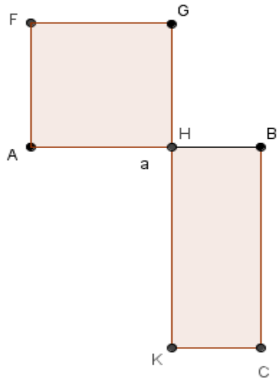
We construct a rectangle ABCD and the middle E of AD. We construct BE and in the extension of DA we take EF=EB and construct the rectangle AFGH. We will prove that the point H is the asking one.



We extend GH, that cuts DC in K. According to the proposal II.6, the rectangle which is defined by DF and AF together with the square of side AE will be equivalent to the square of side EF, ie:

- $DF \cdot AF + EE^2 = EF^2$ (1)
- $DF \cdot AF + AE^2 = EB^2$ (2) as $EF = EB$
- $DF \cdot AF + AE^2 = AE^2 + AB^2$
  (proposal I.47, Theorem of Pythagoras in the triangle EAB)
- $DF \cdot AF = AB^2$
- $(DFGK) = (ABCD)$ (3)

Abstracting from (3) the (AHKD), will have $(AFGH) = (HBCK)$ or $A\ H^2 = \ \cdot B$ or $AH^2 = AB \cdot HB$ or $\dfrac{AH}{HB} = \dfrac{AB}{AH}$



Comments :

In the proposal II.11 the construction takes place in order to divide a given part of a line "α" in such a way that the rectangle with sides "a" and a part of "a", to be equal with the square which have its side the part of "a". This relationship is expressed by $\alpha \cdot (\alpha - b) = b^2$

The proposal II.11 is used in the proposal VI.10.

Proposals of the Elements that are used

- Proposal I.47 (Pythagorean Theorem): In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.
- Proposal II.6: If a straight line is bisected and a straight line is added to it in a straight line, then the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half equals the square on the straight line made up of the half and the added straight line.

Proposition VI.30

To cut a given finite straight line in extreme and mean ratio.

Proof

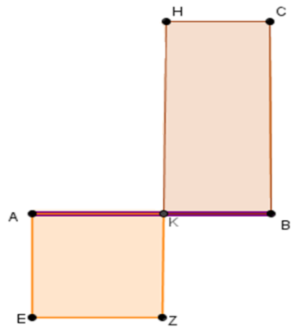
With side "a" we construct a square ABCD (I.46).
We extend DA and construct parallelogram DHZE which is equal to ABCD in a way that the per exaggeration parallelogram AKZE to be similar with the parallelogram ABCD (Proposal VI 29).
As (ABCD)=(DHZE) and if abstracting the parallelogram ADHK that is common to both, then (AKZE)=(BCHK).

But the parallelograms AKZE and BCHK have the same area and equal angles, so according to the proposal VI.14 the sides that contain the equal angles are reversal similar and $\frac{HK}{KZ}=\frac{AK}{KB}$

But $HK = AB$ and $KZ = AK$,

so $\frac{AB}{AK}=\frac{AK}{KB}$.

And as AB> AK it will be AK>KB.



Comments

According to Heath, the construction is an immediate application of the proposal VI.29 in the special occasion that the excess of the parallelogram is a square. This issue combining with the VI.30 is enough proof that this construction is by Euclid.
The proposal VI.30 is used in proposals 1,2,3,4,5,6,7,,8,9,10,11,16,17,18 of XIII Book of Euclid’s Elements.

Proposals that are used:

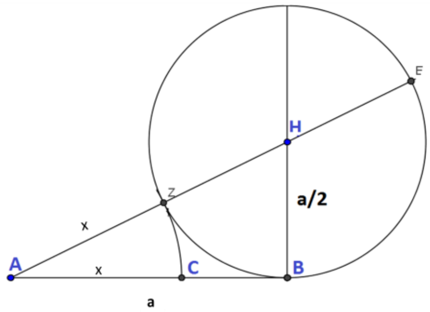
- Proposal I. 46: To describe a square on a given straight line.
- Proposal VI. 14: In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional; and equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.
- Proposal VI. 29: To apply a parallelogram equal to a given rectilinear figure to a given straight line but exceeding it by a parallelogram similar to a given one.

THE PROBLEM IN MORDEN TEXTBOOKS OF CEOMETRY (IN CREECE)

To divide a segment AB, into two unequal segments AC and CB in a way that the biggest of these to be in an extreme and mean ratio to the other and to the AB.

Proof:

Lets have $AB = a$ and a point C with $AC = x$ the largest one. Then $CB = a - x$ and it will be $\frac{AC}{AB} = \frac{CB}{AC}$ (1)
or $AC^2 = AB \cdot CB$
or $x^2 = a \cdot (a - x)$ (2).



The relation (2) can be written
$x^2 + ax - a^2 = 0$ or $x(x + a) = a^2$ (3)

In order to find a point C in the segment AB so that $AC = x$ we draw a circle $\left(H, \frac{\alpha}{2}\right)$ that osculates to the segment AB in the point B and construct AH that cuts the circle to Z and E. Then

$$AB^2 = AZ \cdot AE = AZ \cdot (AZ + ZE) = AZ \cdot (AZ + AB)$$

$$or\ a^2 = AZ \cdot (AZ + a)$$

So, the segment AZ has the right length and the point C will be the sectional of the circle $\left(A, \frac{\alpha}{2}\right)$ and the segment AB.

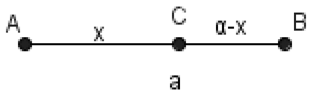
Comment

In these constructions the proposals III.36 και III.37 του III of the Books of Euclid are used.

- Proposition III.36: If a point is taken outside a circle and two straight lines fall from it on the circle, and if one of them cuts the circle and the other touches it, then the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the tangent.
- Proposition III.37: If a point is taken outside a circle and from the point there fall on the circle two straight lines, if one of them cuts the circle, and the other falls on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the straight line which falls on the circle, then the straight line which falls on it touches the circle.

THE PROBLEM AND GEOMETRICALLY SOLVING THE 2nd DEGREE EQUATION –THE NUMDER *Phi (φ)*.

The problem of division of a segment AB from a point C into extreme and mean ratio today is known as Golden Section and the ratio $\frac{AB}{AC} = \frac{AC}{CB}$



or equivalent $\frac{\alpha}{x} = \frac{x}{\alpha - x}$ is called Golden Ratio or Divine Ratio and is symbolized by the greek letter tau (τ). The symbol (τ) means "the cut" or "the section" in Greek. With the problem of Golden Section we can geometrically solve the equation $\alpha(\alpha - x) = x^2$ or $x^2 + \alpha x - \alpha^2 = 0$ which the root is the number $x = \frac{a(\sqrt{5}-1)}{2}$.

Then the golden ratio becomes $\frac{\alpha}{x} = \frac{x}{\alpha - x} = \frac{\sqrt{5}+1}{2}$.

During 19th century, American Mathematician Mark Barr gave to golden ratio a new name *Phi (φ)*.

$$\phi = \frac{\sqrt{5}+1}{2}$$

This symbol *φ* becomes from the name of the Greek sculpture of Classical Ancient Greece Phidias, because we can meet the Golden Ratio to many of its works like Parthenon.
Golden Ratio is an irrational number. Many historical believe that this irrational has been found by the Pythagoreans' about 5th b.C. who believed that irrationals was something like a cosmic error.

Comment

This problem or otherwise the geometrical solution of the equation $x^2 + \alpha x - \alpha^2 = 0$ is described using the proposals III.36 and III.37 of the book III of Euclid's.

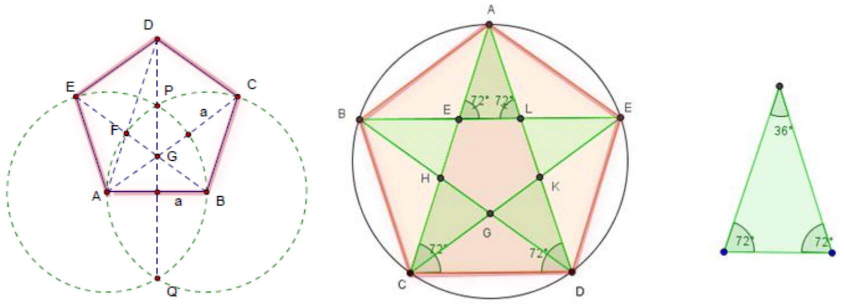
THE PROBLEM IN ANCIENT GREECE

Plato prophesied the importance of the golden ratio long before Euclid describe the details and saw the world in terms of perfect geometric proportions and symmetries, in the forms of the five Platonic solids, the tetrahedron, the cube, the octahedron and the icosahedron. The platonic solids are the only solid that all the seats are regular polygons and each vertex is convex. Each of these solid entered a sphere with all their vertices on it. Every seat in the Platonic solids are regular polygon: equilateral triangle, square, and regular pentagon. The tetrahedron consists of four equilateral triangles, the cube of six squares, the octahedron has eight seats in the form of equilateral triangles, the dodecahedron has twelve regular pentagons as seats and icosahedron twenty equilateral triangles. The dodecahedron and icosahedron are very interest. If any of them constructed with edge length of a unit, it is very easy for someone to recognize the important role played by Colden Ratio to their dimensions.

	Surface area	volume
dodekahedro	$15\varphi/(3-\varphi)$	$5\varphi^3/(6-2\varphi)$
icosahedro	$5\sqrt{3}$	$5\varphi^5/6$

The construction of the pentagon with ruler and compass is very interesting, as shown properties of extreme and mean ratio between the sides and diagonals.

The side and the diagonal of the regular pentagon is also a product of self-similar of the Golden Section.



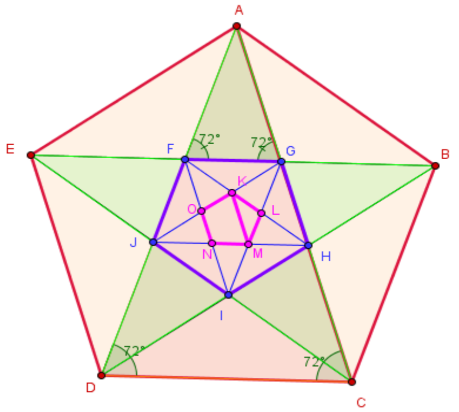
The diagonals of a regular pentagon intersecting around the center of the pentagon, form another smaller regular pentagon, etc. in an infinite sequence of regular pentagons that everyone is into another. The sides of all these pentagons and diagonals of the pentagons are permanently in a golden ratio sequence-based on φ. Each diagonal of the regular pentagon is intersected by another at the site of the golden ratio, which means that each diagonal divides the one that intersects and is divided by it, in extreme and mean ratio. The ratio of any diagonal of a regular pentagon to the side of this pentagon is the golden ratio φ.

$$\frac{FG}{KM} \approx \varphi$$

$$\frac{AB}{FH} \approx \varphi$$

$$\frac{DB}{BC} \approx \varphi$$

$$\frac{BE}{BF} = \frac{AD}{DF} \approx \varphi$$



THE PROBLEM AFTER *EUCLID*

After Euclid problem of division in extreme and mean ratio appears the so-called "Supplement" or Book XIV of the Elements attributed to Hypsicles of Alexandria (2nd BC). Also, it appears in the work of Hero of Alexandria and is related to the determination of the surface of the pentagon and the dodecagon, and the "synagogue" of Pappus of Alexandria in the construction of the icosahedron and dodecahedron and the comparison theorems of their volumes.

In Arab tradition there are indications introduction of the concept of division of a segment in medium and extreme ratio, although the works of some Arabs considered related problems, such as Al-Chouarizmi (about 780-850 AD) and Abul -Ouafa (ca. 940-997 AD).

In the European tradition, the origins of the study of the properties of division in medium and extreme ratio are stemming to Leonardo of Pisa or Fibonacci (ca. 1180- 1250 AD), who examines metric problems of the pentagon and the dodecagon, and the identification problems volume icosahedron and dodecahedron. Fibonacci with the "problem of the rabbits" is creating the following sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... that is called sequence of Fibonacci and is formed according to the rule:

$$u_0 = 1,\ u_1 = 1,\ u_n = u_{n-1} + u_{n-2}$$

The relationship of this sequence with the problem of division in medium and extreme ratio is that the limit of the ratio of the next last term of the sequence is the value of the medium and extreme ratio, ie the root of the equation, $x^2 + \alpha x - \alpha^2 = 0$ ie th $\phi = \frac{\sqrt{5}+1}{2}$ r There are indications that the Fibonacci knew this relationship, which occurs later mathematicians of 16th century, Kepler, Zizar, Simpson.

In 15th-16th century interest in the division in middle and extreme ratio revitalized compared with its applications in geometry and architecture. In this context introduced the term 'golden ratio' by Leonardo da Vinci. In 1509 the project issued "The divine proportion", which although is specially dedicated to the problem of division in "extreme and mean ratio".

CONCLUSIONS

The way that the structure of the division is done, varies between that shown in Figures Euclid and subsequent mathematical as Descartes. For the ancient Greeks the construction of the golden section had not the usefulness of geometric equation solving, but looked more like a constant ratio with application to construct regular polygons in books VI and XIII. The constructions are based on equations and similar areas of rectangles and squares as the product of two line segments is surface, and not straight section as in Descartes. Considering some basic elements of the presence of the 'golden mean', then perhaps understand its greatness. Initially, everywhere eg. in nature, the universe, the human body, in order to impart harmony. Additionally, it is a great discovery for both the science of mathematics and to the development of human history, as it was triggered, so that people begin to perceive the world and life with another eye. That is to feel that they can explain this inexplicable perfection of life and nature through a "divine" number.

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