ABOUT A CIRCLE AND THE ‘POWER’

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ABSTRACT

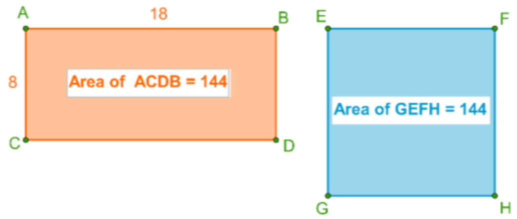
In our project we present the meaning of the term “power” through its historical origins and its progress in a context of Geometry. Initially, we focus on “Euclid’s Elements” and we referring to propositions that are useful in the modern “Geometry of circle” played an important role in developing new terms in 18th century and beyond. Finally, we present and compare proofs of the fundamental theorems involved with the “power of a point” which we meet in Euclidean and Analytic Geometry of Greek secondary schools recommending their teaching through their applications using dynamic geometry tools.

INTRODUCTION

Often in Mathematics studying the terms and definitions of a theory from their historical and etymological origin contribute at their better understanding and highlights important stages of the theory progress. In mathematics the term “power” is widely used and is seemingly different from the meaning that have this term in Physics, in Algebra we meet it as “power of a number”, in Analysis in order to introduce the exponential function and in Geometry as “the power of a point” with respect to a circle. The term “power” in Geometry related with the position of a point and refers to the constant product of two variable amounts that has different interpretations: a) equivalent rectangles, b) constant product of segments of intersecting chords in a circle, c) difference of squares of two real numbers, d) dot product of two vectors.
Different interpretations of the word “power” unless of its real meaning that is to say validity, capacity and strength, appear for commercial purposes resulting opens to interpretations as price, value, for example, expressions such as “power exchange”.

In ancient Greek mathematics the term "power" has a specific mean, this of "value" and "price" but with the meaning of the square. At Diophantus we meet the words «δυναμοδύναμις» ("dinamodinamis"), «δυναμοκύβος» ("dinamokivos") and «κυβοκύβος» ("kivokivos") for the fourth a^4, the fifth a^5 and the sixth a^6 power of a number.

Historically, the first interpretations of the term "power" is "the price of square rectangle" that means the area of a rectangle equals the area of a square. This means that for a reason this was an important transformation of a rectangle to a square.



Gradually gets the mean of the value of a square and later only the meaning of a square. The power of a number, with its current meaning, is probably the extension of the meaning value of square to contain everyone number in one exhibitor.

The term "power of a point with respect to a circle" refers to all rectangles with arbitrary length which transformed to a unique square.

THE MEANING OF "POWER OF A POINT" WITH RESPECT TO A CIRCLE AT EUCLID'S ELEMENTS

The first concept of the "power" in a circle can be found in propositions III.35, III.36 and III.37, that is relevant with the contact or the section of a circle and a line from the III *Book of a Euclid's Elements*. These proposals of *Euclid's Elements* focuses in the relation of rectangles areas which define the segments of chords that defined from their point of intersection.

These proposals of Euclid's *Elements*, the relevant with the notions of power are useful in modern "Geometry of circle", as with their help defined basics concepts such as the power of a point, radical axes and radical centre of three circles that developed in 18^{th} century and beyond.

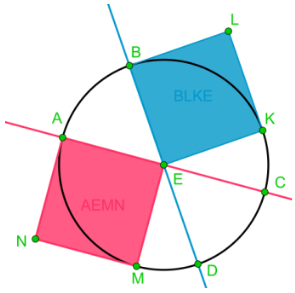
PROPOSITION 35

If in a circle two straight lines cut one another, then the rectangle contained by the segments of the one equals the rectangle contained by the segments of the other".

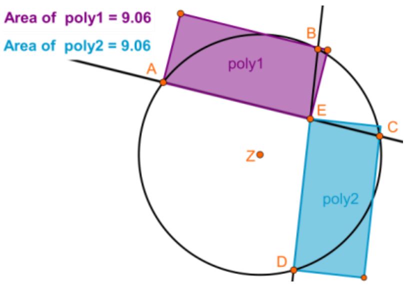
PROOF

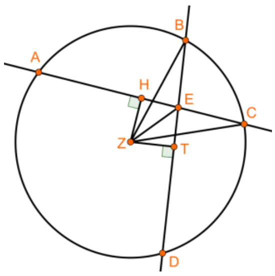
Consider the straight lines which intersect the circle at A, C and B, D respectively and let E be their common point. We are going to prove that the rectangle AE by EC equals the rectangle DE by EB.

First case: If AB and CD passing through the center of the circle ABCD, such that E is the center of the circle, then AE, EC, DE, EB are equal, as radius of the same circle, so the AEMN equals the BLKE.



Second case: Let AC and BD do not pass through the center of the circle and Z is the center. Consider perpendicular lines from Z to AC and BD, be the ZH and the ZT respectively, and also ZB, ZC and ZE.





Because the line ZH (from the center of the circle) cut vertically the AC (chord of the circle), H will be the middle point of the AC, so $AH = HC$.
Because line AC has been cut into equal parts at C and into unequal parts at E, the rectangle AE by EC plus the square of HE equals the square of HC (II.5 “If a straight line is cut into equal and unequal segments, then the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section equals the square on the half.”). We add at both the square of HZ, so the rectangle AE by EC plus the squares of HE and HZ equals the squares HC and HZ. From the Pythagorean theorem we have that the sum of the squares HE and HZ equals the square of ZE and also that the sum of the squares HC and HZ equals the square of ZC. So the rectangle AE by EC plus the square of ZE equals the square of ZC. But ZC equals ZB (as radius of the same circle), so the rectangle AE by EC plus the square of ZE equals the square of ZB. Similar the rectangle DE by EB plus the square of ZE equals the square of ZB. So the rectangle AE by EC plus the square of ZE equals the rectangle DE by EB plus the square ZE. Thus the rectangle AE by EC equals the rectangle DE by EB.

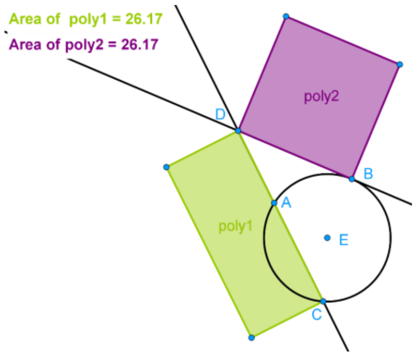
Consequently if in a circle two straight lines cut one another, then the rectangle contained by the segments of the first equals the rectangle contained by the segments of the second.

PROPOSITION 36

"If a point is taken outside a circle and two straight lines fall from it on the circle, and if one of them cuts the circle and the other touches it, then the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the tangent."

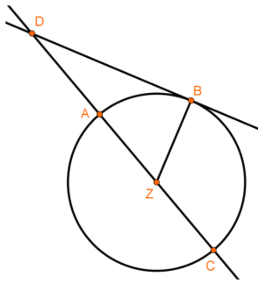
PROOF

Consider a point D outside the circle ABC and from D we take to lines DCA and DB. If DCA cut the circle and DB is tangent of the circle, the rectangle AD by DC equals the square of DB.



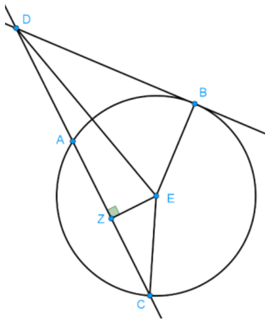
First case: *If DCA pass from the center of the circle.*

Let's say that Z is the center of the circle and we take the ZB so $Z\hat{B}D$ is right angle (III. 18 [14]). Because the line AC has been bisected at Z, so $ZA = ZC$ and CD is added to AC, the rectangle AD by CD plus the square of ZC equals square of ZD (II. 6 [15]). Also ZC equals ZB (as the radius of the same circle). So the rectangle AD by DC plus the square of ZB equals square of ZD. From the Pythagorean Theorem in the right triangle ZBD we have that the square of the ZD equals the square ZB plus the square BD (I. 47 [16]), so the rectangle AD by DC plus the square of ZB equals the sum of squares of ZB and BD. Subtract the square of ZB from each of its. Thus the rectangle AD by DC equals square of tangent BD.



Second case: *If DCA don't pass from the center of the circle.*

Let's say E the center of the circle and consider a perpendicular line from E to AC, the EZ. Also we take EB, EC, ED. The angle EBD is right (III. 18 [14]). And, since a straight line EZ through the center cuts a straight line AC not through the center at right angles,



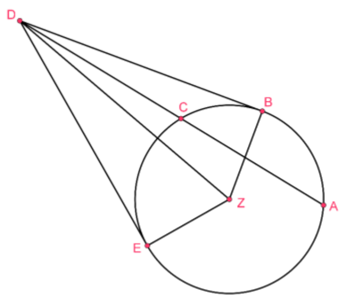
it also bisects it, therefore AZ equals ZC. Since the straight line AC has been bisected at the point Z, and CD is added to it, the rectangle AD by DC plus the square on ZC equals the square on ZD. We add at both the square of ZE. So the rectangle AD by DC plus the squares of ZC and EZ equals the sum of the squares of ZD and ZE. From the Pythagorean Theorem at right triangle EZC we have that the sum of the squares ZC and ZE equals the square of EC. From the Pythagorean Theorem at right triangle EZD we have that the sum of squares of ZD and ZE equals square of ED. So the rectangle AD by DC plus the square of EB equals the square of ED. From the Pythagorean Theorem we have that the sum of squares of EB and BD equals square of ED. So the rectangle AD by DC plus EB equals the sum of the squares of EB and BD. Subtract the square on EB from each. So the rectangle AD by DC equals the square of BD.
Therefore if a point is taken outside a circle and two straight lines fall from it on the circle, and if one of them cuts the circle and the other touches it, then the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the tangent.

PROPOSITION 37

"If a point is taken outside a circle and from the point there fall on the circle two straight lines, if one of them cuts the circle, and the other falls on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the straight line which falls on the circle, then the straight line which falls on it touches the circle."

PROOF

Let a point D outside of the circle ABC. From D we take two lines that fall on the circle ABC, DCA which cut the circle and DB which fall on it. Also the rectangle AD by DC equals square of DB. We will prove that DB touches the circle ABC. We take the tangent of the circle ABC the DE and also we take the centre of the circle, let's say Z. We brought ZE, ZB and ZD. So the $Z\hat{E}D$ is right (III.18).



Because DE is tangent of the circle ABC and DCA cut the circle, the rectangle AD by DC equals the square of DE (III.36). Also the rectangle AD by DC equals the square of DB. So the square of DE equals the square of DB. Therefore DE equals DB. Also ZE equals ZB (as radius of the same circle).
So the two sides DE and ZE equal the two sides DB and ZB, and DZ is the common base of the triangles, therefore the angle $D\hat{E}Z$ equals the angle$D\hat{B}Z$. But the angle $D\hat{E}Z$ is right so the angle $D\hat{B}Z$ is also right. And ZB twice is a diameter.

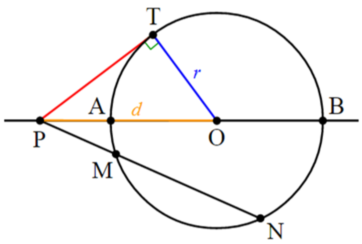
The straight line drawn at right angles to the diameter of a circle, from its end, touches the circle, therefore DB touches the circle ABC. Similarly this can be proved to be the case even if the centre is on AC.
So if a point is taken outside a circle and from the point there fall on the circle two straight lines, if one of them cuts the circle, and the other falls on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the straight line which falls on the circle, then the straight line which falls on it touches the circle.

THE NOTION OF "POWER OF A POINT" WITH RESPECT TO A CIRCLE ACCORDING TO STEINER

The meaning of "power" as equivalence areas don't stop in the Euclid's Elements.
The term "power of a point", with its current meaning introduced from Jacob Steiner. Steiner defines the "power of a point" in1826 at his long article "A Few Geometrical Observations". Steiner displaces the focus of Elements from the length of segments that defines the intersection point of chords of the circle to the same intersection point.
He define the "power of a point" with respect to a specific circle as the number that doesn't depend from the position of the point in the chord but related with the position of the point with respect to that circle.

DEFINITION

The power of a point P with respect to a circle (O, r), as described by Steiner, is the invariant number $|\, d^2 - r^2 \,|$ *, where r is the radius of the circle and d is the PO, and symbolizes with* $\Delta^{P}_{(O,\, r)}$

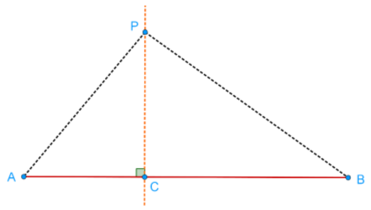


Noted that the power of a point as defined by Steiner, and generally at the begging of its use, was only with absolute value to focus in its constant value for each point. Later it has been used without absolute and takes negatives and positives values to indicate the position of the point with respect to the circle.
This transition from the chords' properties to a point has been important, because in plane geometry the point alone lack of properties. The power of a point isn't a property due to its position at the chord. The power of a point become a relation to point and circle that has a format $a^2 - b^2 = c$ (c constant), with the result that acquires mathematical status. In reality Steiner achieves to connect the definition of power of a point not only with the circle but with generally any geometrical locus of points of the plane that has the property $a^2 - b^2 = c$, where a, b the distance of

the point and two others constant points. The proof of this contention is very easy and is become with the Pythagorean Theorem.

PROOF

Let's say that we have a segment AB, a point P and the perpendicular line to AB from P. Then for any position of point P at the perpendicular line we have:



From the Pythagorean Theorem at right triangle PCA we have:

$PA^2 = PC^2 + AC^2$ or $PA^2 - AC^2 = PC^2$ (1)

From the Pythagorean Theorem at right triangle PCB we have:

$$PB^2 = PC^2 + CB^2$$
$$PB^2 - CB^2 = PC^2 \quad (2)$$

From (1), (2) we have:

$$PA^2 - AC^2 = PB^2 - CB^2$$
$$PA^2 - PB^2 = AC^2 - CB^2$$

Because A, B and C are given $AC^2 - CB^2$ is constant, so $PA^2 - PB^2 = constant.$

The opposite of this contention is direct.

From these we have that the point P which is difference the square of the distances from two constant points A and B is constant and belongs to the segment AB.

THE MEANING OF "POWER OF A POINT" WITH RESPECT TO A CIRCLE AT SCHOOL BOOKS OF GEOMETRY

At Greek school book of Geometry there is a reference to the theorems of the power of the point and its meaning. The power of a point is defined as the subtraction $\Delta^P_{(O,r)} = d^2 - r^2$ with great focus on its property to identify the position of a point with respect to a circle. Also sates and prove the below theorems.

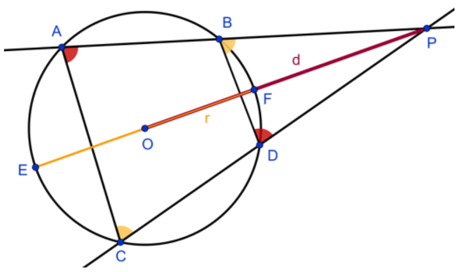
FIRST THEOREM OF THE POWER OF A POINT

Let P, be an arbitrary point inside or outside of a circle (O, r) and secants of the circle AB and CD which intersect at P. Then

$$PA \cdot PB = PC \cdot PD = |d^2 - r^2|$$

PROOF

First case: *Point P is outside of the circle.*



The triangle PBD is similar with the triangle PAC because:

For ABCD we have:

- The angle $P\hat{B}D$ equals to the angle $P\hat{C}A$, because $P\hat{C}A$ is one of the angles of the inscribed quadrilateral and $P\hat{B}D$ is the opposite outer corner.
- The angle $P\hat{D}B$ equals to the angle $P\hat{A}C$, because $P\hat{A}C$ is one of the angles of the inscribed quadrilateral and $P\hat{D}B$ is the opposite outer corner.

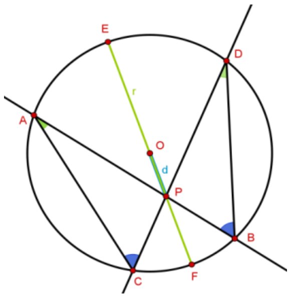
So $\frac{PB}{PC} = \frac{PD}{PA} = \frac{AC}{BD}$ or $\frac{PB}{PC} = \frac{PD}{PA}$ or $PA \cdot PB = PC \cdot PD$

If OP = d, the radius of the circle and PO intersect the circle in F, E we have:

$PA \cdot PB = PC \cdot PD = PE \cdot PF$ or $PA \cdot PB = PC \cdot PD = (d + r)(d - r)$

$$PA \cdot PB = PC \cdot PD = d^2 - r^2 = \Delta^P_{(O,r)}$$

Second case: *Point P is inside of the circle*



The triangle PBD is similar with the triangle PAC because:

- The angle $P\hat{B}D$ equals to the angle $P\hat{C}A$, because inscribed angle $P\hat{C}A$ views the same arc (AD) with the inscribed angle $P\hat{B}D$.
- The angle $P\hat{D}B$ equals to the angle $P\hat{A}C$, because inscribed angle $P\hat{A}C$ views the same arc (BC) with the inscribed angle $P\hat{D}B$.

So $\frac{PB}{PC} = \frac{PD}{PA} = \frac{AC}{BD}$ or $\frac{PB}{PC} = \frac{PD}{PA}$

$$PA \cdot PB = PC \cdot PD$$

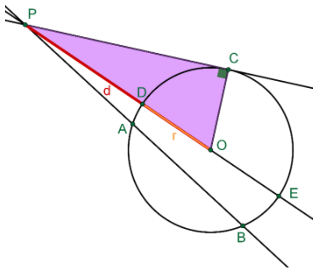
Set OP = d, r the radius of the circle and PO intersect the circle in F, E we have:

$$PA \cdot PB = PC \cdot PD = PE \cdot PF$$
$$PA \cdot PB = PC \cdot PD = (d + r)(d - r)$$
$$PA \cdot PB = PC \cdot PD = d^2 - r^2 = \Delta^P_{(O,r)}$$

SECOND THEOREM OF THE POWER OF A POINT

Assume P an arbitrary point outside of a circle (O, r), a line from P that intersect the circle at A, B and PC tangent of the circle then:

$$PC^2 = PA \cdot PB = |d^2 - r^2|$$



PROOF

We take the PO, which cut the circle at D and E, and PO = d. From the first theorem of the power of a point we have

$$PA \cdot PB = PD \cdot PE$$
$$PA \cdot PB = (d - r)(d + r)$$
$$PA \cdot PB = d^2 - r^2 \quad (1)$$

From the Pythagorean Theorem at the right rectangle PCO we have

$$PO^2 = PC^2 + OC^2$$
$$PC^2 = PO^2 - OC^2$$

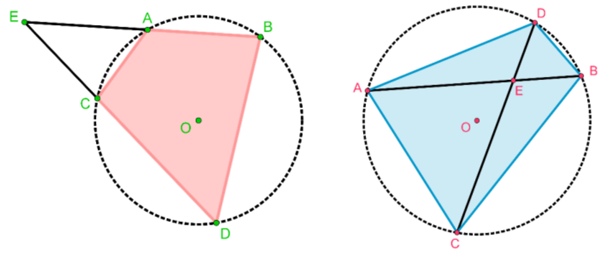
$$PC^2 = d^2 - r^2 \quad (2)$$

From (1) and (2) we have: $$PC^2 = PA \cdot PB = d^2 - r^2$$
$$PC^2 = PA \cdot PB = |d^2 - r^2| = \Delta^P_{(O,r)}$$

APPLICATION OF THE NOTION OF "POWER OF A POINT"

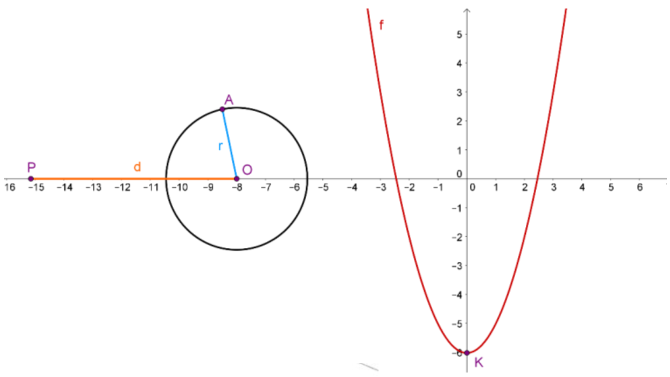
The power of a point with respect to a circle has many different uses.
With the power of a point, as we previously said, we can find the position of one point with respect to a circle. So we have the following cases:

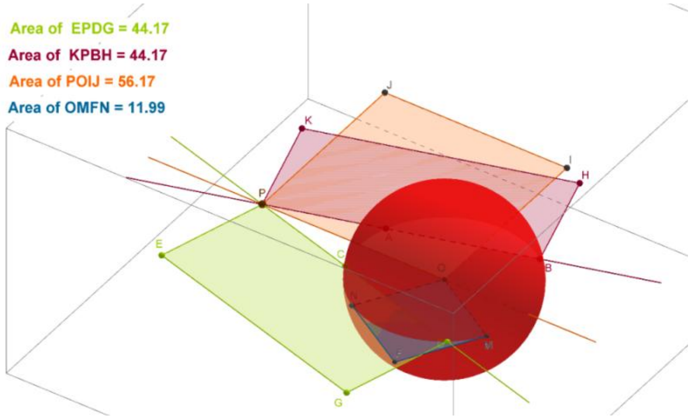
- If $\Delta^P_{(O,r)} > 0$ then P is outside point of the circle ($d^2 > r^2$)
- If $\Delta^P_{(O,r)} < 0$ then P is inside point of the circle ($d^2 < r^2$)
- If $\Delta^P_{(O,r)} = 0$ then P is point of the circle ($d^2 = r^2$)



Also with the power of a point we can check if there exist a circle passing through four points. So we have that if for every four points A, B, C and D such that AB intersect with CD at P and it's true that $PA \cdot PB = PC \cdot PD$ then the A, B, C and D are in the same circle. Also as a result with the power of a point theorem we can prove that a quadrilateral is inscribable.

We can also see the power of a point as a function. If we have a point P in a given line which pass from the centre of a circle and we take the function of the power of the point as it moves through line so to take $f(d) = d^2 - r^2$ where d the distance of the point and the centre of the circle and r the radius of the circle, is a parabola, as r^2 is just a constant. We can see that the top of parabola is for $d^2 = 0$ happens when d coincides the centre of the circle so $f(d) = -r^2$



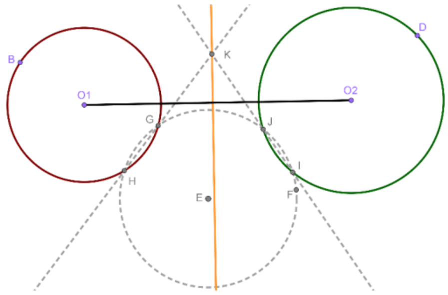


Notion of the power of a point can be extended with respect not only to a circle but also to a sphere

or any algebraic curve.
With the help of the power of a point we could define the radical axis and the radical center.

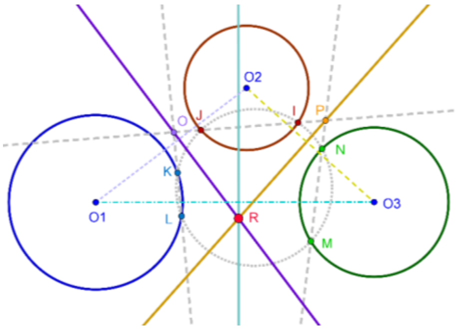
RADICAL AXES

If we take the circles: C_1 (O_1, R_1) and C_2 (O_2, R_2). The geometric locus of points that have equal powers with respect to two circles is a straight line vertical to the centerline O_1O_2, called the radical axis of the two circles.



To find the radical axis of two circles C_1 (O_1, R_1) and C_2 (O_2, R_2), we consider a circle C (O, R) that intersects the two others. Radical axis of the two circles is the vertical line from the common chords' point of intersection to the centerline. If two circles intersect in two points A and B, then the radical axis of the two circles is the straight line that set by the two points.

RADICAL CENTER



Assume now that we have in the same plane three circles: C_1 (O_1, R_1), C_2 (O_2, R_2) and C_3 (O_3, R_3). The radical axes *(set by circles two by two)* are passing through the same point called the radical center of the three circles. To find the radical center of the three circles C_1 (O_1, R_1), C_2 (O_2, R_2) and C_3 (O_3, R_3) we consider a circle C (O, R) that intersect the three others. We consider, after that, the radical axes which will be intersected in the radical center.

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